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# ON CERTAIN CLASSES OF MEROMORPHICALLY P-VALENT STARLIKE FUNCTIONS

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**Abstract.** Let  $M_{n+p-1}(\alpha)$  denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \dots + a_{k+p-1}z^k + \dots \quad (p \in N = \{1, 2, 3, \dots\})$$

that are regular in the annulus  $D = \{z : 0 < |z| < 1\}$  and satisfy

$$\operatorname{Re} \left\{ \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} \right\} < -\alpha$$

for  $0 \leq \alpha < p$  and  $|z| < 1$ , where

$$D^{n+p-1}f(z) = \frac{1}{z^p} \left( \frac{z^{n+2p-1}f(z)}{(n+p-1)!} \right)^{(n+p-1)}.$$

We prove that  $M_{n+p}(\alpha) \subset M_{n+p-1}(\alpha)$ , where  $n$  is any integer greater than  $-p$ .

We also consider some integrals of functions in the class  $M_{n+p-1}(\alpha)$ .

## 1. Introduction

Let  $\Sigma_p$  denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \dots + a_{k+p-1}z^k + \dots, \quad (1.1)$$

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which are regular in the annulus  $D = \{z : 0 < |z| < 1\}$ , where  $p$  is a positive integer. The Hadamard product or convolution of two functions  $f$  and  $g$  in  $\Sigma_p$  will be denoted by  $f * g$ . Let

$$D^{n+p-1}f(z) = \frac{1}{z^p(1-z)^{n+p}} * f(z), \quad (z \in D) \quad (1.2)$$

or, equivalently,

$$\begin{aligned} D^{n+p-1}f(z) &= \frac{1}{z^p} \left( \frac{z^{n+2p-1}f(z)}{(n+p-1)!} \right)^{(n+p-1)} \\ &= \frac{1}{z^p} + (n+p)a_0 \frac{1}{z^{p-1}} + \frac{(n+p+1)(n+p)}{2!} a_1 \frac{1}{z^{p-2}} + \dots \\ &\quad \dots + \frac{(n+k+2p-1)\dots(n+p)}{(k+p)!} a_{k+p-1} z^k + \dots \quad (z \in D), \end{aligned}$$

where  $n$  is any integer greater than  $-p$ .

In this paper, among other things, we shall show that a function  $f(z)$  in  $\Sigma_p$ , which satisfies one of the conditions

$$\operatorname{Re} \left\{ \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} \right\} < -\alpha \quad (z \in U = \{z : |z| < 1\}), \quad (1.3)$$

where  $n$  is any integer greater than  $-p$ , is meromorphically  $p$ -valent starlike in  $U$ . More precisely, it is proved that, for the classes  $M_{n+p-1}(\alpha)$  of functions in  $\Sigma_p$  satisfying (1.3),

$$M_{n+p}(\alpha) \subset M_{n+p-1}(\alpha) \quad (0 \leq \alpha < p). \quad (1.4)$$

Since  $M_0(\alpha)$  equals  $\Sigma^*(\alpha)$  (the class of meromorphically  $p$ -valent starlike functions of order  $\alpha$  in  $U$  [4]), the starlikeness of members of  $M_{n+p-1}(\alpha)$  is a consequence of (1.4).

## 2. Properties of the class $M_{n+p-1}(\alpha)$

In proving our main results, we shall need the following lemma due to Jack [3].

**Lemma.** *Let  $w$  be non-constant regular in  $U = \{z : |z| < 1\}$ ,  $w(0) = 0$ . If  $|w|$  attains its maximum value on the circle  $|z| = r < 1$  at  $z_0$ , we have  $z_0 w'(z_0) = kw(z_0)$ , where  $k$  is a real number,  $k \geq 1$ .*

**Theorem 1.**  $M_{n+p}(\alpha) \subset M_{n+p-1}(\alpha)$  for each integer  $n$  greater than  $-p$ .

**Proof.** Let  $f(z) \in M_{n+p}(\alpha)$ . Then

$$\operatorname{Re}\left\{\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)}\right\} < -\alpha. \quad (2.1)$$

We have to show that (2.1) implies the inequality

$$\operatorname{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} < -\alpha. \quad (2.2)$$

Define  $w(z)$  in  $U$  by

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = -\frac{p + (2\alpha - p)w(z)}{1 + w(z)}. \quad (2.3)$$

Clearly,  $w(z)$  is regular and  $w(0) = 0$ . Using the identity

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - (n+2p)D^{n+p-1}f(z), \quad (2.4)$$

the equation (2.3) may be written as

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{n+p + (n+3p-2\alpha)w(z)}{(n+p)(1+w(z))}. \quad (2.5)$$

Differentiating (2.5) logarithmically, we obtain

$$\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} = -\frac{p + (2\alpha - p)w(z)}{1 + w(z)} + \frac{2(p - \alpha)zw'(z)}{(1 + w(z))(n + p + (n + 3p - 2\alpha)w(z))}. \quad (2.6)$$

We claim that  $|w(z)| < 1$  in  $U$ . For otherwise (by Jack's lemma) there exists  $z_0$  in  $U$  such that

$$z_0 w'(z_0) = k w(z_0), \quad (2.7)$$

where  $|w(z_0)| = 1$  and  $k \geq 1$ . The equation (2.6) in conjunction with (2.7) yields

$$\frac{z_0 (D^{n+p} f(z_0))'}{D^{n+p} f(z_0)} = -\frac{p + (2\alpha - p)w(z_0)}{1 + w(z_0)} + \frac{2(p - \alpha)kw(z_0)}{(1 + w(z_0))(n + p + (n + 3p - 2\alpha)w(z_0))}. \quad (2.8)$$

Thus

$$\operatorname{Re} \left\{ \frac{z_0 (D^{n+p} f(z_0))'}{D^{n+p} f(z_0)} \right\} \geq -\alpha + \frac{p - \alpha}{2(n + 2p - \alpha)} \geq -\alpha, \quad (2.9)$$

which contradicts (2.1). Hence  $|w(z)| < 1$  in  $U$  and from (2.3) it follows that  $f(z) \in M_{n+p-1}(\alpha)$ .

**Theorem 2.** Let  $f(z) \in \Sigma_p$  satisfy the condition

$$\operatorname{Re} \left\{ \frac{z (D^{n+p-1} f(z))'}{D^{n+p-1} f(z)} \right\} < -\alpha + \frac{p - \alpha}{2(c + p - \alpha)} \quad (z \in U). \quad (2.10)$$

for a given integer  $n > -p$  and  $c > 0$ . Then

$$F_c(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t) dt \quad (2.11)$$

belongs to  $M_{n+p-1}(\alpha)$ .

**Proof.** Let  $f(z) \in M_{n+p-1}(\alpha)$ . Define  $w(z)$  in  $U$  by

$$\frac{z (D^{n+p-1} F_c(z))'}{D^{n+p-1} F_c(z)} = -\frac{p + (2\alpha - p)w(z)}{1 + w(z)}. \quad (2.12)$$

Clearly,  $w(z)$  is regular and  $w(0) = 0$ . Using the identity

$$z (D^{n+p-1} F_c(z))' = c D^{n+p-1} f(z) - (c + p) D^{n+p-1} F_c(z), \quad (2.13)$$

the equation (2.12) may be written as

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = -\frac{p + (2\alpha - p)w(z)}{1 + w(z)} + \frac{2(p - \alpha)zw'(z)}{(1 + w(z))(c + (c + 2p - 2\alpha)w(z))}. \quad (2.14)$$

We claim that  $|w(z)| < 1$  in  $U$ . For otherwise (by Jack's lemma) there exists  $z_0$  in  $U$  such that

$$z_0 w'(z) = kw(z_0), \quad (2.15)$$

where  $|w(z_0)| = 1$  and  $k \geq 1$ . Combining (2.14) and (2.15), we obtain

$$\frac{z_0(D^{n+p-1}f(z_0))'}{D^{n+p-1}f(z_0)} \geq -\alpha + \frac{p - \alpha}{2(c + p - \alpha)} \geq -\alpha, \quad (2.16)$$

which contradicts (2.10). Hence  $|w(z)| < 1$  in  $U$  and from (2.12) it follows that  $F(z) \in M_{n+p-1}(\alpha)$ .

Similarly, from Theorem 2, we have

**Corollary.** *Let  $f(z) \in M_{n+p-1}(\alpha)$ . Then  $F_c(z)$  defined by (2.11) belongs to the class  $M_{n+p-1}(\alpha)$ .*

**Remarks.** (1). A result of Bajpai[1] turns out to be a particular case of the above Theorem 2 when  $p = 1, n = 0, \alpha = 0$  and  $c = 1$ .

(2). For  $p = 1, n = 0$  and  $\alpha = 0$ , the above Theorem 2 extends a result of Goel and Sohi[2].

**Theorem 3.** *Let  $f(z) \in M_{n+p-1}(\alpha)$ . Then  $F_{n+p}(z)$  defined by (2.11) with  $c = n + p$  belongs to the class  $M_{n+p}(\alpha)$ .*

**Proof.** For the function  $F_{n+p}(z)$  defined by (2.11) with  $c = n + p$ , we have

$$cD^{n+p-1}f(z) = (n + p)D^{n+p}F_{n+p}(z) - (n + p - c)D^{n+p-1}F_{n+p}(z) \quad (2.17)$$

Taking  $c = n + p$  in the above relation (2.17), we obtain

$$D^{n+p-1}f(z) = D^{n+p}F_{n+p}(z). \quad (2.18)$$

This implies that  $F_{n+p}$  belongs to the class  $M_{n+p-1}(\alpha)$ .

**Theorem 4** *Let  $F_c(z) \in M_{n+p-1}(\alpha)$  and let  $f(z)$  be defined as (2.11). Then  $f(z) \in M_{n+p-1}(\alpha)$  in  $|z| < R_c$ , where*

$$R_c = \frac{-(p - \alpha + 1) + \sqrt{(p - \alpha + 1)^2 + c(c + 2(p - \alpha))}}{c + 2(p - \alpha)}. \quad (2.19)$$

**Proof.** Since  $F_c(z) \in M_{n+p-1}(\alpha)$ , we can write

$$\frac{z(D^{n+p-1}F_c(z))'}{D^{n+p-1}F_c(z)} = -(\alpha + (p - \alpha)u(z)), \quad (2.20)$$

where  $u(z) \in P$ , the class of functions with positive real part in  $U$  and normalized by  $u(0) = 1$ . Using the equation (2.13) and differentiating (2.20), we obtain

$$-\frac{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} + \alpha}{p - \alpha} = u(z) + \frac{zu'(z)}{(c + p) - (\alpha + (p - \alpha)u(z))}. \quad (2.21)$$

Using the well known estimates,  $\frac{|zu'(z)|}{\operatorname{Re}u(z)} \leq \frac{2r}{1-r^2} (|z| = r)$  and  $\operatorname{Re}u(z) \leq \frac{1+r}{1-r} (|z| = r)$ , the equation (2.21) yields

$$\operatorname{Re}\left\{-\frac{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} + \alpha}{p - \alpha}\right\} \geq \operatorname{Re}u(z)\left\{1 - \frac{2r}{(1 - r^2)(c + p - (\alpha + (p - \alpha)\frac{1+r}{1-r}))}\right\}. \quad (2.22)$$

Now the right hand side of (2.22) is positive provided  $r < R_c$ . Hence  $f(z) \in M_{n+p-1}(\alpha)$  for  $|z| < R_c$ .

## References

1. S.K. Bajpai, A note on a class of meromorphic univalent functions, *Rev. Roumaine Math. Pures Appl.* **22**(1977), 295-297.
2. R.M. Goel and N.S. Sohi, On a class of meromorphic functions, *Glas. Mat.* **17**(1981), 19-28.
3. I.S. Jack, Functions starlike and convex of order  $\alpha$ , *J. London Math. Soc.* (2) **3**(1971), 469-474.
4. V. Kumar and S.C. Shukla, Certain integrals for classes of  $p$ -valent meromorphic functions, *Bull. Austral. Math. Soc.* **25**(1982), 85-97.
5. St. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.* **49**(1975), 109-115.

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